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Introduction to Dynamic Programming

I recommend you to work in groups of two. Groups of three are generally discouraged. Use of Matlab is required. Your solutions have to be handed to me by the *9am of the day* we correct the exercise in class. You have to hand *both* a copy of the program (if required) and a copy of the written solutions. Try to be as streamlined as possible in writing. You should submit just one copy per *group*. The name of the members in the group should appear at the top of the solution handed. *Each* member of the group should be ready to stand to correct the exercise in class. Each group should prepare a slide with the Matlab code used to obtain the results. I would like you to discuss the structure of the program, not the specific commands used. The schedule of the course will be as follows:

1.

2. Thursday, **12 September** 14:45-16:15
   Lecture 1, *Practical Dynamic Programming*

3. Thursday, **19 September** 14:45-16:15
   Lecture 2: *Some Mathematical Foundations*

4. Thursday, **26 September**, 14:45-16:15
   Solution of *Exercise 1*

5. Friday, **27 September**, 16:45-18:15
   Lecture 3, *Business Cycles, some background*

6. Friday, **4 October**, 16:45-18:15
   Solution of *Exercise 2*

7. Thursday, **10 October**, 14:45-16:15
   Lecture 4, *RBC model*

8. Friday, **11 October**, 16:45-18:15
   Lecture 5, *Solving the RBC model by Value Function Iteration*

9. Tuesday, **14 November**, 14:45-16:15
   Solution of *Exercise 3*

Literature

1. Ljungqvist Lars and Thomas Sargent (2018) **Recursive Macroeconomic Theory**, Chapter 3 and 4.

2. Stokey Nancy and Robert Lucas with Edward Prescott (1989) **Recursive Methods in Economics Dynamics**, chapter 2 and 3.

Exercise 1
Dynamic programming with analytical results

1) *Deterministic Dynamic Programming* A consumer maximizes his intertemporal logarithmic utility with respect to the sequence of consumptions $\{c_s\}_{s=0}^{\infty}$, capital $\{k_s\}_{s=1}^{\infty}$, subject to the constraint that $k_{t+1} = A(k_t)^{\alpha} - c_t$ (what is the depreciation rate?). That is

$$\max_{\{c_s\}_{s=0}^{\infty},\{k_s\}_{s=1}^{\infty}} \sum_{s=0}^{\infty} \beta^s \ln(c_s),$$
$$s.t. \quad k_{t+1} = A(k_t)^{\alpha} - c_t$$

where $0 < \beta < 1$, $0 < A$, $0 < \alpha < 1$.

- Show that the production function and the utility function satisfy the standard assumptions:

  $$f'(\cdot) > 0, f''(\cdot) < 0, u'(\cdot) > 0, u''(\cdot) < 0.$$

  Define the Inada conditions on production and utility. Are they satisfied here?

- Write the Bellman equation corresponding to this problem in such a way that the transition equation $x_{t+1} = g(x_t, u_t)$ does not depend on the state variable x_t. That is

  $$V(k) = \max_{k'} \ \ln(Ak^{\alpha} - k') + \beta V(k')$$

  Use the Envelope theorem to state the Benveniste and Scheinkman condition corresponding to this problem.

  Write also the first order condition corresponding to the Bellman equation. (Hint: consider k_{t+1} as your control variable)

- (Value function iteration: illustration). We now show how the 'value function iteration' method can be applied to this case. Start with $V_0(k) \equiv 0$ and solve the one period problem. Show that the solution is to choose $k' = 0$. Show then that $V_1(k) = \ln(A) + \alpha \ln(k)$. If we solve the one period problem we find $k' = \frac{\beta\alpha}{1+\beta\alpha} Ak^{\alpha}$ and $V_2(k) = \ln(\frac{A}{1+\beta\alpha}) + \beta\ln(A) + \alpha\beta\ln(\frac{\beta\alpha A}{1+\beta\alpha}) + \alpha(1+\alpha\beta)\ln(k)$. If we were proceeding further we would find that at the *nth* iteration the policy function $h_n(k)$ is such that

  $$h_n(k) = k' = \frac{\alpha\beta\sum_{s=0}^{n}(\alpha\beta)^s}{1+\alpha\beta\sum_{s=0}^{n}(\alpha\beta)^s} Ak^{\alpha},$$

  while the value function $V_n(k)$ satisfies

  $$V_n(k) = \alpha\sum_{s=0}^{n}(\alpha\beta)^s \ln(k) + A_n$$

where A_n is a sequence of constant independent of k such that

$$\lim_{n\to\infty} A_n = (1-\beta)^{-1}\left\{\ln\left[A\left(1-\beta\alpha\right)\right] + \frac{\beta\alpha}{1-\beta\alpha}\ln(A\beta\alpha)\right\}.$$

Find then the limit value function defined by $V(k) = \lim_{n\to\infty} V_n(k)$ and the limit policy function defined by $h(k) = \lim_{n\to\infty} h_n(k)$.

- (Guess and verify method) We now show how the 'guess and verify' method can be applied to this case. We guess that the solution to the Bellman equation takes the form

  $$V(k) = E + F\ln k.$$

  Show then that the policy function satisfied by this value function takes the form

  $$k' = h(k) = \frac{\beta F}{1+\beta F}Ak^{\alpha}.$$

  Find the value of E and F that satisfies the functional equation (This method is also called the method of undetermined coefficients.)

- (Policy function iteration: illustration). We now show how the 'Policy function iteration' method can be applied to this case. Guess a policy function of the form

  $$k' = h_0(k) = Gk^{\alpha}$$

  for some arbitrary constant G. Then form

  $$V_{h_0}(k_0) = \sum_{s=0}^{\infty}\beta^s\ln(Ak_s^{\alpha} - k_{s+1}).$$

  Then choose $k' = h_1(k)$ by maximizing

  $$\ln(Ak^{\alpha} - k') + \beta V_{h_0}(k').$$

  Show, using the previous point, that this algorithm converges to the optimal policy function in one step. Then calculate

  $$V_{h_1}(k_0) = \sum_{s=0}^{\infty}\beta^s\ln(Ak_s^{\alpha} - k_{s+1}).$$

  What about the convergence of $V_{h_n}(k_0)$?

2) *Stochastic Dynamic Programming* Let E_0 indicates the expected value taken with respect the information at time 0 and consider the following optimization problem:

$$\max_{\{c_s\}_{s=0}^{\infty},\{k_s\}_{s=1}^{\infty}} E_0 \sum_{s=0}^{\infty} \beta^s u(c_s), \qquad 0 < \beta < 1,$$

subject to

$$\begin{aligned}
u(c_s) &= \ln c_s, \\
f(k_t,\theta_t) &= A\theta_t\,(k_t)^{\alpha}\,, \quad 0<\alpha<1, \\
k_{t+1} &= f(k_t,\theta_t) - c_t, \\
\ln\theta_t &= \epsilon_t, \\
& \{\epsilon_t\} \quad \textit{iid and normal} \;\; N(0,1).
\end{aligned}$$

Assume that at time t when the agent chooses the stock of capital k_{t+1}, θ_t is known while θ_{t+1} is not.

- What is the economic meaning of the random variable θ_t? In particular what is its economic interpretation?
- Write the Bellman equation corresponding to this problem in such a way that the transition equation $x_{t+1} = g(x_t, u_t)$ does not depend on x_t. (Hint: use k_{t+1} as a control variable). Use the Envelope theorem to calculate the derivative of the value function.

  Write also the first order condition corresponding to the Bellman equation.
- Let E_t and f_1 indicate the expected value conditional to the information available at time t and the marginal productivity of capital respectively. Show that in equilibrium the following condition holds

$$1 = \beta E_t\left[\frac{c_t}{c_{t+1}}\; f_1(k_{t+1},\theta_{t+1})\right] \tag{1}$$

- Verify that the value function for this problem takes the following form

$$V(k,\theta) = V_1 + V_2 \ln k + V_3 \ln\theta$$

  for some (to-be-determined) coefficients V_1, V_2 and $V_{3.}$ In particular show that

$$\begin{aligned}
V_1 &= (1-\alpha\beta)^{-1}\left[\ln A + (1-\alpha\beta)\ln(1-\alpha\beta) + \alpha\beta\ln\alpha\beta\right](1-\beta)^{-1}\,, \\
V_2 &= (1-\alpha\beta)^{-1}\alpha, \\
V_3 &= (1-\alpha\beta)^{-1}.
\end{aligned}$$

- Show then that the optimal policy function takes the form

$$k_{t+1} = h(k_t,\theta_t) = \alpha\beta A k_t^{\alpha}\theta_t.$$

- Use this result to show that the log of the capital stock follows a first-order stochastic difference equation (an $AR(1)$ process). Is this process stationary? Show then that the process for output is exactly the same as that for k_t, except for a multiplicative (or additive in logs) shift.

- (Policy function iteration: illustration). We now show how the 'Policy function iteration' method can be applied to this case.

  Iteration 1: Guess a policy function of the form

  $$k' = h_0(k, \theta) = Gk^{\alpha}\theta$$

  for some arbitrary constant G. Then calculate the value of teh problem under policy h_0:

  $$V_{h_0}(k_0, \theta_0) = E_0 \sum_{s=0}^{\infty} \beta^s \ln(Ak_s^{\alpha}\theta_s - k_{s+1}).$$

  Iteration 2: Choose $k' = h_1(k, \theta)$ by maximizing

  $$\ln(Ak^{\alpha}\theta - k') + \beta E V_{h_0}(k', \theta),$$

  and then calculate

  $$V_{h_1}(k_0, \theta_0) = E_0 \sum_{s=0}^{\infty} \beta^s \ln(Ak_s^{\alpha}\theta_s - k_{s+1}).$$

  Subsequent iterations: Continue iterating on this scheme until successive h_j have converged. In particular show that, in this example, the algorithm converges to the optimal policy function in one step. What about the convergence of $V_{h_n}(k_0, \theta_0)$?

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Exercise 2
Solving Bellman equations by value function iteration

1) *Deterministic Dynamic Programming* Take a canonical Ramsey model as discussed in class. The production function is

$$y = f(k) = A\,k^{\alpha}$$

where A is a constant. The utility function is

$$u(c) = \ln(c)$$

The resource constraint is

$$c + k' = (1-\delta)\,k + f(k)$$

The problem of the agent is:

$$v(k) = \max_{k',c} \{u(c) + \beta\, v(k')\}$$
$$\text{s.t. } c \geq 0, \quad k' \geq 0, \quad c + k' = (1-\delta)\,k + f(k)$$

or alternatively:

$$v(k) = \max_{k'} \{u((1-\delta)\,k + f(k) - k') + \beta\, v(k')\} \tag{2}$$

subject to $k' \geq 0$ and $(1-\delta)\,k + f(k) - k' \geq 0$. Assume $\delta = 1$.

- We know that $k' = g^k(k) = \alpha\beta A k^{\alpha}$ is the exact policy function of this problem. Use the values $\alpha = 0.3$, $\beta = 0.95$ and $A = 7.30376$ to find the steady state value of k and y. Plot the policy function for values of k in the range $[0, 60]$.
- We are going to solve the same problem numerically, as if the closed-form solution for the policy function g^k was unknown. To do so discretize the support for k such that $k \in \{k_1, k_2, ...k_N\}$. In particular set $k_0 = 0$ and $k_N = 60$ and consider equally spaced interval. Exclude k equal to zero as a possible (initial) state—i.e. k_1 is strictly greater than zero. Assume that for whatever initial k you are considering the agent can choose the next period k only among the given k_i, $i = 1, 2..N$. Rule out all values of k' that imply a negative value for today consumption. Solve the problem in (2) by using value function iteration. Assume that the initial guess for the value function is the zero line, $V_0(k) = 0$, $\forall k$.

- Plot in the same graph the exact policy function g^k with the one found in the previous point for $N = 50$, $N = 500$ and $N = 1000$. What can you say about the discretization method?

2) *Stochastic Dynamic Programming* We now consider a Ramsey problem with shocks. The production function is given by:

$$y = z\,f\,(k) = z\,A\,k^{\alpha}$$

where $z \in \{z_b, z_g\}$ is a (productivity shock) which evolves according to the following transition function:

$$\Gamma\,(z, z') = \begin{pmatrix} \gamma_{gg} & \gamma_{bg} \\ \gamma_{gb} & \gamma_{bb} \end{pmatrix} = \begin{pmatrix} 0.90 & 0.10 \\ 0.10 & 0.90 \end{pmatrix} \tag{3}$$

where γ_{ij} is the probability that if you are currently in state i you will end up in state j in the next period. Set $z_b = 0.98$ and $z_g = 1.02$.

- Write down the Bellman equation associated with this problem, when the control variable is k'. The state variables are k and z_i, $i = b, g$. Write the Bellman equation using the summatory sign $\sum$.
- Assume that $\delta = 1$, $\alpha = 0.3$ and $\beta = 0.95$ as in the previous exercise and solve the model numerically as if the policy function $g(k, z)$ was unknown. To do so discretize the support for k such that $k \in \{k_1, k_2, ...k_N\}$. In particular set $k_0 = 0$ and $k_N = 60$ and consider equally spaced interval. Exclude k equal to zero as a possible (initial) state—i.e. k_1 is strictly greater than zero. Assume that for whatever initial k the agent can choose the next period k only among the given k_i, $i = 1, 2..N$. Rule out all values of k' that imply a negative value for today consumption. Solve the Bellman equation derived in the previous point by use of value function iteration. Assume that the initial guess for the value function is the zero line, $V_0(k, z_i) = 0$, $\forall k, z$. Set $N = 500$ and plot the policy functions for each of the two possible values of z, $z_b = 0.98$ and $z_g = 1.02$, together with the policy function obtained in the previous exercise.
- Now assume that $\delta = 0.15$. The other parameters are identical to those of the previous point. Solve the model again. Plot the policy functions for each of the two possible values of z, $z_b = 0.98$ and $z_g = 1.02$, together with the policy function obtained in the previous point for the corresponding two values of z. Try to explain the effects of the reduction of δ on the policy function. Give clear economic intuition.
- Simulate the economy for 5,500 periods. Drop the first 500 observations and calculate the standard deviation of consumption.

- Propose a metric to evaluate the cost of business cycles in terms of consumption equivalent losses in steady state consumption. Are the costs big? Now, assume that

$$u\left(c\right)=\frac{c^{1-\sigma}-1}{1-\sigma}$$

  Solve the problem again for different values of σ. How does the cost of business cycles varies with σ? From this exercise, would you conclude that the welfare costs of business cycle fluctuations are large or small?

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Exercise 3
Solving the RBC model

- Use data from FRED

  https://fred.stlouisfed.org/

  to calculate the HP-filtered ($\lambda = 1600$) standard deviations of the quarterly time series of the logarithm of $\{y_n, h, c, i, w\}$ where y_n is logged labor productivity (Nonfarm Business Sector: Real Output Per Hour of All Persons (OPHNFB)), h is logged aggregate hours worked per capita (Indexes of Aggregate Weekly Hours of Production and Nonsupervisory Employees, Total Private (AWHI)), c is logged consumption expenditures (Personal consumption expenditures per capita (A794RC0Q052SBEA)), i is logged investment expenditures per capita (for example Real Gross Private Domestic Investment (GPDIC1)) and w are logged wages nonfarm business sector real compensation per hour. For the US take for example the series Population (POPTHM). Download the series for the entire post WWII sample period. Discuss the properties of the data for the entire post WWII period and for the period starting in the early 90's. Do you see important differences?

- Consider the real business cycle model discussed in class characterized by the following social planner problem:

  $$V(k,z) = \max_{k',h} u\left[(1-\delta)k + e^z f(k,h) - k', 1-h\right] + \beta E\left[V(k',z')\right] \quad (4)$$

  subject to

  $$\begin{aligned} 0 &\leq k' \\ 0 &\leq (1-\delta)k + e^z f(k,h) - k' \\ z' &= \rho z + \epsilon \end{aligned}$$

  where k is today capital stock, ϵ is iid normal with standard deviation σ_ϵ, $\epsilon \sim N(0, \sigma_\epsilon^2)$ and the functional forms are as follows:

  $$\begin{aligned} u(c, 1-h) &= (1-\alpha)\ln c_t + \alpha \ln(1-h_t) \\ f(k,h) &= Ak^\theta h^{1-\theta} \end{aligned}$$

  The parameter values are θ=0.4, $\rho = 0.95$, $\sigma_\epsilon = 0.017$, $\delta = 0.012$, $\beta = 0.99$ and $\alpha = 0.64$. How would you choose the value of A? Could you choose it to target a steady state capital-consumption (or capital-output) ratio roughly equal to 2?

- Solve the model by value function iteration after discretizing the AR(1) process for aggregate productivity. How many grid point for z are you using? How many grid point for k are you using? Why is it optimal to choose just a few grid points for z (say no more than 5)?

- Simulate the economy for 5,500 periods. Drop the first 500 observations. Apply the HP filter with $\lambda = 1600$, report statistics and compare with the data.

- How does the fit of the model changes when you focus on the subperiods starting from the early 90's? Which features of the data explain the change in fit?

- How would you modify the model to better explain the most recent features of the data?