Luiss, Fall 2018
ROME Course:
Mathematical Methods for Economics and Finance
First Part: prof. Fausto Gozzi
Detailed Program

Textbook: Carl P. Simon and Lawrence E. Blume: "Mathematics for economists".

Note: in the references SB will denote the textbook above.
Chapters from 1 to 12, Chapters 26-27-28-29 and Chapters A1-A2-A4-A5 in SB are supposed to be known from basic undergraduate courses in mathematics. The material from Chapters 6-7-8-9-10-11-26-27-28-A4 has been reviewed in the precourse. The material of the precourse will not be repeated during the course.

Week 1: Introduction to the course. Introduction to several variables functions *(Sept. 12-13).*
Lectures
- Organization of the course and its website.
- Functions from R^n to R^m. Basic definitions: domain, law, maximal domain, image, graph. Components when m>1. Special case m=1. Examples.
- Curves as functions from R to R^n. Image of a curve and examples.
- Inner product, orthogonality, distance, norm, Carnot Theorem (with idea of proof) and its geometric interpretation.
- Concept of equation of a geometrical object. Three main types of equations (Cartesian, Implicit, Parametric: example of the lines).
- Space generated (Span) by a set of vectors. Vector subspaces and affine subspaces.
- Parametric equations of lines and planes in R^n. Parametric equations of lines as images of affine curves.
- Lines in R^2, their three types of equations (cartesian, implicit, parametric) and how to pass from one of them to the other.
- Equations of half lines and segments.
- Exercises: finding an equation of a line passing through two given points, of a line passing through a given point and orthogonal (or parallel) to another given line. Also find an equation of the line passing through a given point and orthogonal (parallel) to a plane.
- Linear equations in several variables and their geometric interpretations as equations of hyperplanes (in particular lines in R^2 and planes in R^3). Geometrical meaning of the coefficients and examples. Geometric interpretation of the solutions of linear systems as intersections of hyperplanes.

- Subsets of R^2 and R^n and their graphical representation.
- Basic topological properties of sets (neighborhoods, interior, exterior, boundary; closed, open bounded, compact, convex sets). Examples on the slides with, in particular, discussion on the convexity of D.

- Problem: it is hard to represent the graph of functions of several variables. Hence we try to represent them using lower dimensional objects: level sets (dimension n-1) and restrictions (dimension 1).
- Level curves/surfaces/sets. Definition and examples.

Exercises

Find equations of lines and planes which satisfy given properties (passing through 2/3 pts, parallel, orthogonals to given vectors, etc.).
Graphic representation of subsets of R^2 (possibily R^3), e.g. maximal domains, finding their topological properties.
Graphic representation of level curves of simple functions (e.g. x^2 + y^3, y^2+2x, etc).
Graphic representation of the images of simple curves (e.g. (2cos t, 3sin t), (t^2,t^3) etc).

Week 2: Basic calculus for several variables *(September 19-20)*.

Lectures

Restrictions of functions to lines and to curves. Definition through composition of functions. Examples.
Functions from R^n to R^m and operations on them (sum, product, quotient, composition). Brief discussion on composition and its domain and examples.
Elementary functions.
Properties of functions: boundedness, montonicity along lines (example: utility increasing along rays), convexity/concavity (through epigraph and through inequalities with precise explanation of their meaning); max/min and sup/inf.
Local properties. In particular local boundedness and local extremal points. Example: x^2 is locally bounded in each point of R but is not globally bounded.

Limits: definitions (also limit to infinity).
Continuity and Theorem on continuity of elementary functions and the operations on them.
Weierstrass Theorem (WT) with two examples.
Coercive functions: definition.
Variant of WT: domain A closed not bounded and f coercive. Examples with calculus of the limit using "separated variables" and comparison theorems. Two theorems on this and example on how to use them, in particular using suitable inequalities.

Partial and directional derivatives, gradient: definitions and examples
Theorem on partial differentiability (or C^1 or C^2) of elementary functions and their combinations.
Theorem: if a function is C^1 then every directional derivative (in the direction v) is found taking the inner product of the gradient with v.
Directions of increase and decrease.
Corollary: all direction with positive (negative) inner product with the gradient are of increase (decrease).
Proof of the fact that the gradient "points" in the direction of maximal increase.

Differential (graphical idea) with informal definition of tangent plane.
Definition of differential as linear map. Theorem of total differential.
Tangent plane/hyperplane and its cartesian equation. Examples.
Tangent plane is below/above the graph in the case of convex/concave functions.
Basic inequality for it and its use to prove that critical points are global extremes.
Monotonicity along directions.
Second partial derivatives, Hessian Matrix, Schwarz Theorem,
Jacobian matrix, tangent vector to a curve and its geometric interpretation.
Regular Curves: definition and examples where a is a non regular point: the tangent line to the image may or may not exist.
Theorem: Jacobian of a composition.
Special case: derivative of a function along a curve.
Examples.

Exercises
Exercises on composition of functions and its derivatives
Finding restrictions, finding partial and directional derivatives from gradient or directly.
Find if a direction is of increase or decrease using gradient or directly.
Compute gradient, Hessian, Jacobian, tangent vector to given functions
Find equations of lines, planes, hyperplanes, tangent to graphs of given functions.
Prove coercivity of given functions using separation or comparison.
Prove existence of maxima/minima.

References for the first 2 weeks
Slides of the course.
SB Chapters:
- 12 (Sections from 12.2 to 12.6)
- 13-14 (All)
- 30 (Weierstrass Theorem: 30.1)

Week 3: Implicit Functions Theorem *(September 26-27)*.
Lectures

- Implicit functions: motivations and examples.
- Implicit function theorem in R^2 (Dini Theorem) with proof.
- Tangent vector to a curve. Recall regular curves in R^2.
- Restatement of The Dini Theorem to say that, on the points where the gradient is nonzero, the level sets of a function G:R^2 to R are image of regular curves: their tangent vector can be computed and it is orthogonal to gradG.
- Implicit Function Theorem in R^n, again interpretation with level sets and their tangents;
- Systems of implicit functions: general theorem in the nonlinear case with comparison with the linear case.
- Examples of economic applications.

Exercises
- exercises on implicit function theorem.
- Apply the implicit function theorem in various cases also in applications. In particular to find how extremal points very with parameters.
- Find tangents to level sets. Comparative statics.
See the exercises on the slides and on the homework.

References
Slides of the course.
SB Chapters:
- Chapter 15 except section 15.5 and 15.6. (Section 15.6 is interesting to read).

Week 4: Unconstrained Optimization ***(October 3-4).***
Lectures

- Formulation of general optimization problems (i.e. finding global extremals). Example: utility or profit maximization.
- Constrained and Unconstrained optimization (whether the set B is open or not open). Example where the boundary is big (a circle).
- First order necessary conditions: Fermat's Theorem (Thm 38) with proof. Examples.
-The set of candidates is the union of three parts: critical points (I_1), boundary points (I_2), nondifferentiability points (I_3). In dimension 2 or more the boundary points are infinitely many. In the unconstrained case only I_1 and I_3 are present.
- Examples of computation of critical points. One easy and two where the set of critical points is infinite in dimension 2 and 3.

- Second order necessary condition and second order sufficient conditions: main idea using Taylor expansion up to the order 2, first in dimension 1 and then in dimension n>1. The problem is to find the sign of the quadratic form associated to the second derivative.

- Quadratic forms and symmetric matrices. Definition and Proposition on the one-to-one correspondence between them. Definition of the sign of them.
- Theorem on the spectrum of symmetric matrices (Theorem 23.16): it is real, the matrix is always diagonalizable and the eigenvector can be chosen orthogonal, so the diagonalizing matrix P can be chosen such that the transpose is equal to the inverse.
- Theorem: relationship between the sign of a symmetric matrix and the sign of the eigenvalues.
- Theorem: sign of symmetric matrices using NW principal minors and using all principal minors (Theorems 16.1 and 16.2 in the book). Examples where we compute the sign of symmetric matrices in dimension 2-3-4.

- Theorem: second order necessary conditions (17.6) and second order sufficient conditions (17.2). The information given by the theorem: choosing between 3 alternatives: local min, local max, saddle. Examples where the information is full and where is only partial or none. Analysis of some simple cases through level curves or inequalities to prove that given critical points are or not global extremals: $x^2 + y^2$, $xy(1-xy)$.

- Finding global extremals in the unconstrained case: three methods.
(a) Using Definition and suitable inequalities.
(b) Using a proposition on restrictions to find sup and inf and to deny that a point is global extremal: first discussion and an example like: $x^3 + y^2$; here, restricting to the line $x=0$ we immediately see that there are no global extremals and that inf is $-\infty$ and sup is $+\infty$.
(c) Using theorems connecting convexity/concavity and critical points and the characterization of convexity on open convex sets using the sign of the Hessian matrix. More precisely:
- Theorem: every critical point of a convex (concave) function is global min (max) pt. Proof.
- Theorem: On a given open convex set A a C^2 function is convex (concave) if and only if D^2f is SDP (SDN) on A.
Examples: $x^2 \log y$, $x \log y$, $x^\alpha y^{(1-\alpha)}$, $x^\alpha y^{(1-\alpha)}-x-y$, ecc.).

Exercises
Sign of symmetric matrices looking at the spectrum or at the minors. Also in parametric cases.
Unconstrained optimization problems: find candidates (in particular critical points) and apply second order conditions, where possible.
Find, if they exist, global max/min pt in the unconstrained case, in various examples.
Find inf and sup of given functions.
See the exercises on the slides and on the homework

References
Slides of the course.
SB Chapters:
- Chapters 16-17 (except 16.3).
- Chapter 23: Sections 23.7 and 23.8

Weeks 5: Constrained Optimization.
Lectures (October 10-11)
- Problem: find global maxima and minima in the general constrained case. Applied examples: utility maximization (standard and with leisure) and cost minimization, (SB examples 18.1, 18.2, 18.3).
- Review of the main methods recalling what said for the case of unconstrained optimization.
- Standard form of constrained optimization problems. Standard form in the examples.
- Necessary conditions of first order (Lagrange multipliers Theorem) for equality constraints. Graphic illustration in the case of on equality constraint.
Constraint Qualifications (NDCQ) in the case of equality constraints (SB Theorems 18.1 and 18.2).
Lagrangian function.

- Example (minimization of costs) where we write the Lagrange system. Other examples.
Solving exercises using the restriction to the set B reducing the problem to a one variable max/min problem. In this case one has to find where the one dimensional variable varies and then go back to the starting problem. Example where, to find where the one dimensional variable varies, one has to solve another, simpler, constrained optimization problem.

- Lagrange multiplier theorem in the case of inequality constraints. NDCQ in this case.
- Graphical illustration in the case of one constraint and in the case of the triangle.
- Example: utility maximization.
- Lagrange multiplier theorem in the mixed case. NDCQ in this case.
- Idea and reference on second order conditions.

- Finding the global extremals in constrained optimization problems: methods (definition, restrictions, convexity/concavity)
(a) - Use of definition with inequalities example also looking at the graph of level curves. Use of definition when we know existence by Weierstrass: example of cost minimization. Examples of the use of Weierstrass Theorem (standard or with coercivity) to find existence of global extrema.
(b) - use of restrictions as in the unconstrained case. Also use of restrictions to the boundary. Examples.
(c) - Recall the definition of convex/concave functions and characterization of C^1 convex/concave functions using tangent hyperplanes.
- Theorem: if a function is convex/concave over a convex set B then critical points (by Lagrange Theorem) with negative/positive multipliers lambda are global minimum/maximum points.
- Uniqueness theorem in the case of strict convexity/concavity (recalling definition of convex/concave functions).
- How to prove that the set B is convex when it is defined, as usual, as intersection of sublevel/level sets? Theorem stating that sublevel/upperlevel sets of convex/concave functions are convex and level sets of linear functions are convex.
- Definition of quasiconvex/quasiconcave function. Theorem: compositions of monotonic transformations (defined in section 20.3) and of convex/concave functions are quasiconvex/quasiconcave.
- Theorem stating that if g is quasiconvex and h linear then B is convex using that the intersection of convex sets is convex. Proof.
- Examples of application with Cobb-Douglas function.
- Theorem 21.22: if a function is convex/concave over a convex set B then critical points (by Lagrange Theorem) with negative/positive multipliers lambda are global minimum/maximum points.

Exercises
- Various examples where we check NDCQ and find the candidate points (e.g. example 18.5, 18.6 and 19.9 in SB).
- Find global extremals of functions in the constrained case with equality, inequality or mixed constraints.
- Prove that a given set is compact or not compact

Weeks 6: Envelope Theorem and Economic Applications
Lectures (October 17-18)

Problem: given a parametric optimization problem, find how the extremal points and the extremal values vary as function of the parameters.
Using IFT to find how the extremal points vary with respect to parameters: example (a polynomial depending on a parameter).
Value Function (VF) and Envelope Theorem (ET). Use ET to show how the VF varies as functions of the parameters. Examples (e.g. Indirect Utility Function and Marshallian Demand). Interpretation of the multiplier.

- Fritz-John Theorem and various types of Constraint Qualifications

- Homogeneous functions and Euler Theorem.

Exercises
- Applications of envelope theorems to find derivatives of the value function. Checking assumptions of envelope theorems.
- Show if a given function is quasiconvex/quasiconcave using sublevel/upperlevel sets or composition of monotonic transformations and convex/concave functions.
- Prove that a given set is convex or not convex.
- Exercises where we check NDCQ and find the candidate points (e.g. example 18.5, 18.6 and 19.9 in SB).
- A problem with production and pollution and on the optimal tax rate to keep the pollution below a given level.
- Exercises on parametric constrained optimization.

References for weeks 5 and 6.
Slides of the course.
SB Chapters:
- 18 (Sections 18.1, 18.2, 18.3, 18.4, 18.5, 18.7)
- 19 (Sections 19.2, 19.4; read also 19.1).
- 20 (Section 20.1)
- 21 (Sections 21.1, 21.2, 21.5 and small part of 21.3 and 21.4: see the above program).
- 22 (Read Sections 22.1 and 22.2)
- 30 (Sections 30.1, 30.2, 30.3, 30.4).